SPECIAL CHARACTERISTICS OF FLOW OF VISCOUSLY FREE-FLOWING MEDIA IN TUBES

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UDC 539.37

§ 1. We consider the plane fully established motion of a viscously free-flowing medium in a slit with a length $L\,(0 \le x \le L)$ and a width $2a\,(-a \le y \le a)$, at whose ends the uniformly distributed pressures p_1 and p_2 are given. In the absence of mass forces, the equations for the stresses have the form

$$\partial\sigma_x/\partial x + \partial\sigma_{xy}/\partial y = 0,\ \partial\sigma_{xy}/\partial x + \partial\sigma_y/\partial y = 0. \tag{1.1}$$

It is assumed that the motion takes place only along a slit along the x axis. Then from the equation of continuity it follows that v_x is a function of the coordinate y. For a viscously free-flowing medium, the components of the stress tensor in the plane case are connected by the relationship [1, 2]

$$(\sigma_x - \sigma_y)^2 + 4\sigma_{xy}^2 = \sin^2\varphi\left[\sigma_x + \sigma_y + 2\,\mathrm{ctg}\,\varphi k - \frac{2\mu}{\sin\varphi}\frac{\partial v_x}{\partial y}\right]^2, \tag{1.2}$$

where φ is the angle of internal friction; k is the coefficient of adhesion; μ is the viscosity. In the case of isotropic deformation, the main components of the tensor of the stresses and the rates of deformation coincide, which leads to the dependence [1]

$$\frac{2\sigma_{xy}}{\sigma_x - \sigma_y} = \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\Big/\frac{\partial v_x}{\partial x}. \tag{1.3}$$

Since $\partial v_x/\partial x = 0$, from (1.3) follows

$$\sigma_x = \sigma_y = p. \tag{1.4}$$

Relationships (1.1), (1.2), and (1.4) make up a closed system of equations for determination of the unknown stresses and the velocity of the flow v_x. By virtue of symmetry with respect to the axis $y = 0$, we shall seek the solution in the region $y > 0$.

Taking account of equality (1.4), from (1.2) we obtain an expression for the tangential stress

$$\sigma_{xy} = p\sin\varphi + k\cos\varphi - \mu\partial v_x/\partial y. \tag{1.5}$$

Solving Eq. (1.5) simultaneously with the equations of motion (1.1) and the condition of adhesion at the wall of the slit, we find the functions of the normal and tangential stresses and the velocities of the flow v_x in the form

$$p = C_0(x - y\sin\varphi) + C_1; \tag{1.6}$$

$$v_x = \frac{C_0\cos^2\varphi}{2\mu}(y - a)(y - C_2); \tag{1.7}$$

$$\sigma_{xy} = [C_0(x - y\sin\varphi) + C_1]\sin\varphi + k\cos\varphi - C_0\cos^2\varphi[y - (a + C_2)/2], \tag{1.8}$$

where C_0, C_1, C_2 are some constants, which will be determined below.

From the symmetry of the problem and the continuity of the tangential stresses it follows that the motion of a viscously free-flowing medium in a limitingly stressed state cannot take place over the whole width of the slit. In actuality, in the contrary case, with $y = 0$, $\sigma_{xy} = 0$, which contradicts equality (1.8). From this it follows that, at the center of the slit, an elastic core with a width $2y_0$ is formed, moving like a solid body. At the boundary of the elastic core, the derivative of the velocity $\partial v_x/\partial y$ reverts to zero. Then from (1.7) it follows that

$$2y_0 = a + C_2. \tag{1.9}$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 159-164, November-December, 1977. Original article submitted August 16, 1976.

It is obvious that, with a uniformly distributed pressure over the ends in the initial and final sections of the slit, there must be formed transitional regions of the formation of the elastic core.

To determine the unknown constants entering into the solution (1.6)-(1.8) we make the following assumptions. The friction forces in the sections of the formation of the liquid core, as well as the dimensions of these sections, can be neglected in comparison with the length of the slit. The mean value of the component of the stress σ_x in the boundary cross sections of the region of the forming one-dimensional flow in the elastic core and in the zone of the motion of the viscously free-flowing medium is equal to the pressures p_1 and p_2 at the ends of the slit.

Then, from (1.6), it follows that

$$C_0 = \frac{p_2 - p_1}{L}, \; C_1 = p_1 + C_0 \frac{a + y_0}{2} \sin \varphi.$$

We determine the width of the elastic core from the condition of the equality of the stresses at the boundary of the core y_0 and the difference of the forces applied at the ends:

$$(p_1 - p_2)\, y_0 = \int_0^L \sigma_{xy \mid y=y_0} dx = \left[\frac{p_1 + p_2}{2} L - \frac{p_1 - p_2}{2} (a - y_0) \sin \varphi\right] \sin \varphi + k \cos \varphi \cdot L.$$

From this last equality we finally obtain

$$y_0 = \frac{[(p_1 + p_2) L - (p_1 - p_2)\, a \sin \varphi] \sin \varphi + 2k \cos \varphi \cdot L}{(p_1 - p_2)(1 + \cos^2 \varphi)}. \tag{1.10}$$

Thus, the width of the elastic core with a given length of the slit depends not only on the pressure drop but also on the values of the pressures at the end of the slit. Here there is a considerable difference in the flow of viscously free-flowing media with internal friction, compared to the motion of viscous and viscoplastic liquids. With given pressures p_1 and p_2, the equality (1.10) enables us to determine the maximal value of the length of the slit L, for which the natural inequality $y_0 \leq a$ is satisfied:

$$L \leqslant \frac{2a (p_1 - p_2)}{(p_1 + p_2) \sin \varphi + 2k \cos \varphi}.$$

For narrow slits, assuming $a/L \ll 1$, we obtain

$$y_0 = \frac{L \sin \varphi}{1 + \cos^2 \varphi} + \frac{2 (p_2 \sin \varphi + k \cos \varphi) L}{(p_1 - p_2)(1 + \cos^2 \varphi)}. \tag{1.11}$$

From relationships (1.10), (1.11) it follows that, with small pressure drops or with sufficiently long slits, there is no motion of the viscously free-flowing medium and the tube is closed. This latter circumstance is explained by the linear dependence of the tangential stresses on the coordinate x. Knowing the width of the elastic core, from (1.7), (1.9) we can obtain an expression for the mass flow rate Q of the viscously free-flowing medium through unit width of the slit:

$$Q = 2\left[\int_{y_0}^{a} v_x(y)\, dy + v_x(y_0)\, y_0\right] = \frac{p_1 - p_2}{\mu L} \cos^2 \varphi\, (a - y_0)^2 \frac{2a + y_0}{3}.$$

The case of the flow of viscous and viscoplastic liquids is obtained from the solution found with $\varphi = 0$ and $k = 0$; $\varphi = 0$, respectively.

§ 2. In the presence of mass forces of gravity the equations of motion in the case of radial symmetry in cylindrical coordinates have the form [3]

$$\begin{gathered} \partial\sigma_z/\partial z + \partial\sigma_{rz}/\partial r + \sigma_{rz}/r = \gamma, \\ \partial\sigma_r/\partial r + \partial\sigma_{rz}/\partial z + (\sigma_r - \sigma_\theta)/r = 0, \end{gathered} \tag{2.1}$$

where γ is the specific weight of the medium.

Assuming that only the coordinate v_z differs from zero, from the equation of continuity we obtain $\partial v_z/\partial z = 0$. Then the equation of a limitingly stressed state and the relationship connecting the tensors of the stresses and the deformation rates are written in the following manner [1, 3]:

$$(\sigma_z - \sigma_r)^2 + 4\sigma_{rz}^2 = \sin^2 \varphi \left(\sigma_r + \sigma_z + 2k \operatorname{ctg} \varphi - \frac{2\mu}{\sin \varphi} \frac{\partial v_z}{\partial r}\right)^2; \tag{2.2}$$

$$\frac{2\sigma_{rz}}{\sigma_z - \sigma_r} = \frac{\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}}{2\frac{\partial v_z}{\partial z}}. \tag{2.3}$$

From Eq. (2.3) and the equality $\partial v_z/\partial x = 0$ it follows that $\sigma_r = \sigma_z$. We shall seek the solution of the problem in the form

$$\sigma_r = \sigma_z = \sigma_\theta = p. \tag{2.4}$$

In this case, the component of the stress σ_θ satisfies the inequality proposed in [3]:

$$\frac{\sigma_r + \sigma_z}{2}(1 - \sin\varphi) < \sigma_\theta < \frac{\sigma_r + \sigma_z}{2}(1 + \sin\varphi).$$

Taking account of (2.4), for the tangential stress σ_{rz} from (2.2) we obtain

$$\sigma_{rz} = p\sin\varphi + k\cos\varphi - \mu\partial v_z/\partial r. \tag{2.5}$$

With the condition of adhesion of the viscously free-flowing medium at the wall of the tube, the solution of the system of Eqs. (2.1), (2.4), and (2.5) has the form

$$\begin{gathered} p = p_2 = \text{const}, \\ v_z = C_1 \ln(r/R) + (F/\mu)(r - R) - (\gamma/4\mu)(r^2 - R^2), \\ \sigma_{rz} = \gamma r/2 - \mu C_1/r, \end{gathered} \tag{2.6}$$

where $F = p_2 \sin\varphi + k\cos\varphi$. With $\varphi \to 0$, the solution (2.6), in distinction from the plane case, does not go over into the solution for a viscoplastic medium [4], since, with $\varphi = 0$, there is a change in the type of the starting system of equations. Since, in the region of a limitingly stressed state, the pressure p is constant, at one end of the tube there may not be a transitional region of the formation of an elastic core.

Let us examine the case of the absence of a transitional section at the outlet of the tube. Then the constant p_2 is equal to the pressure in the external medium with $z = L$. In the inlet section we neglect the tangential stresses in the elastic region in comparison with the friction forces at the walls of the tube and assume that the mean pressure in the initial cross section of the core is equal to the pressure p_1 at the inlet of the tube. We denote the radius of the elastic core by r_0.

At the boundary r_0, the relationship

$$\left.\frac{\partial v_z}{\partial r}\right|_{r=r_0} = \frac{C_1}{r_0} + \frac{F}{\mu} - \frac{\gamma}{2\mu} r_0 = 0$$

is satisfied, from which we obtain

$$C_1 = -\frac{r_0}{\mu}F + \frac{\gamma}{2\mu}r_0^2.$$

We find the value of r_0 from the condition of the equality of all the forces acting on the core of the flow

$$(p_1 - p_2)\pi r_0^2 + \pi r_0^2 \gamma L = 2\pi r_0 F L.$$

We finally obtain

$$r_0 = 2F/[(p_1 - p_2)/L + \gamma]. \tag{2.7}$$

From the latter equality it can be seen that, in spite of the different forms of the solution, the expression for the boundary of the core r_0 with $\varphi = 0$ coincides with the case of a viscoplastic liquid [4].

In the derivation of relationship (2.5) for the tangential stresses σ_{rz} it was assumed that the flow takes place in the positive direction of the z axis and that the velocity of the flow rises toward the center of the tube, i.e., $\partial v_z/\partial r \le 0$ with $r_0 \le r \le R$.

Then, taking account of Eq. (2.7), from the expression for the velocity of the flow (2.6) we obtain

$$\frac{\partial v_z}{\partial r} = -\frac{F}{\mu}\frac{r_0}{r} + \frac{\gamma}{2\mu}\frac{r_0^2}{r} + \frac{F}{\mu} - \frac{\gamma}{2\mu} r \leqslant 0.$$

We divide both parts of the latter inequality by the expression $1 - r_0/r$:

$$F \leqslant (\gamma/2)(r + r_0).$$

The inequality obtained must be satisfied for the whole region of the flow of the viscously free-flowing medium with $r_0 \le r \le R$ and consequently

$$F \leqslant \gamma r_0. \tag{2.8}$$

Substituting into (2.8) the expression for the radius of the elastic core, we obtain

$$(p_1 - p_2)/L \leqslant \gamma. \tag{2.9}$$

Thus, fully established motion of a viscously free-flowing medium along a vertical round tube is possible only in the case where the pressure gradient set up by the external forces at the ends does not exceed the specific weight of the medium. From inequality (2.9), specifically, it follows that, in the absence of mass forces ($\gamma = 0$), motion without breaking down the condition of adhesion at the wall is completely impossible; i.e., under the action of the forces applied at the ends, a viscously free-flowing medium is either at rest (there is closing of the tube) or it moves with slippage at the walls.

We obtain a second limiting inequality from expression (2.7) and the condition $r_0 < r$:

$$(p_1 - p_2)/L > (2F - \gamma R)/R. \tag{2.10}$$

We transform the right-hand part of (2.10) in the following manner:

$$(2F - \gamma R)/R = (2\pi RF - \pi R^2\gamma)/\pi R^2.$$

The expression $F_1 = 2\pi RF$ represents the possible friction force at the wall of the tube without taking account of the viscous component, and $F_2 = \pi R^2\gamma$ is the weight of the medium arriving at a unit of length. If $F_1 < F_2$, inequality (2.10) is satisfied for any arbitrary pressure and only the sole limiting relationship remains (2.9). In the contrary case, the simultaneous satisfaction of inequalities (2.9) and (2.10) is required, which follows a two-sided limitation on the length of the tube:

$$(p_1 - p_2)/\gamma \leqslant L < (p_1 - p_2)/(2F/R - \gamma).$$

It can be shown that the latter inequality is not contradictory; i.e.,

$$(p_1 - p_2)/\gamma < (p_1 - p_2)/(2F/R - \gamma). \tag{2.11}$$

Actually, from (2.8) we obtain

$$\gamma > F/R > 2F/R - \gamma,$$

from which follows the validity of the inequality (2.11)

In conclusion we note that, for the case of nonisotropic deformation [1], the qualitative picture of the flow of a viscously free-flowing medium in a tube does not change. The authors wish to express their thanks to A. Kh. Mirzadzhanzade for his evaluation of the work and his useful observations.

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